Elementary Wavelet Analysis and an Application to the Study of $H^1(\mathbb{R}^n)$

by

A. G. Marasingha

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

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Signature of Author ....
Department of Mathematics
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Certified by ....
T J Lyons
Wallis Professor of Mathematics
Thesis Supervisor

Accepted by ....
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Abstract

A discussion of orthonormal wavelet bases and wavelet series, utilising the notion of multiresolution approximations. The regular wavelets of compact support are seen to form an unconditional basis of the space $H^1(\mathbb{R}^n)$ of Stein and Weiss.

Thesis Supervisor: T J Lyons
Title: Wallis Professor of Mathematics

Contents

Chapter 1

Introduction

In this article, I will present an overview of elementary wavelet analysis and conclude by using wavelet methods to characterize the real Hardy space $H^1(\mathbb{R}^n)$.

What are wavelets? At the definition level, they are simply a basis (usually orthonormal) of $L^2(\mathbb{R})$ with certain properties which express their regularity, oscillatory nature (hence 'wave') and localization in space (hence '-let'). Wavelet analysis can be viewed as a monumental improvement on Fourier analysis.

The primary goal of this article is to construct and discuss the properties of orthonormal wavelet bases and wavelet series. A fundamental tool in this process is the multiresolution approximation, which, as the name suggests, is way of describing a given $L^2(\mathbb{R}^n)$ function at different resolutions. In fact the concept was developed in connection with image processing, in which the interpretation is direct.

We will see that any multiresolution approximation gives rise to a wavelet basis of $L^2(\mathbb{R}^n)$ via a 'scaling function' ϕ. In particular, we will demonstrate how to find multiresolution approximations which lead to wavelets (and scaling functions) of compact support.

Finally, it will be demonstrated that the wavelets of compact support are just what we need to form an unconditional basis of the space $H^1(\mathbb{R}^n)$ of Stein and Weiss.

Chapter 2

Fourier analysis

Though wavelet analysis is in many ways superior to Fourier analysis, Fourier methods are indispensable in the development of the wavelet theory. In this section, I will state the results which will be needed in later sections. Though I will only look at Fourier analysis on the real line, analogous results hold in $\mathbb{R}^n$, and I will use such results later with neither statement nor proof.

2.1 Fourier Integrals

Definition 2.1 *If $f \in L^1(\mathbb{R})$ then its Fourier transform is, by definition:*

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} e^{-i\xi x} f(x) dx$$

Theorem 2.2 *If $f \in L^1(\mathbb{R})$ then*

1. *$\hat{f} \in L^\infty(\mathbb{R})$ and $\|\hat{f}\|_\infty \leq \|f\|_1$;*
2. *$\hat{f}$ is uniformly continuous on $\mathbb{R}$;*
3. *if f' exists and $f' \in L^1(\mathbb{R})$, then $\widehat{f'}(\xi) = i\xi \hat{f}(\xi)$; and*
4. *$\hat{f}(\xi) \to 0$, as $\xi \to \pm\infty$.*

I will frequently have occasion to call upon the following simple but useful formula:

Remark 2.3 *If $f \in L^1(\mathbb{R})$ and $g(x) := \frac{x-b}{a}$, for some constants a and b, then $f \circ g \in L^1(\mathbb{R})$ and*

$$\widehat{f \circ g}(\xi) = ae^{-ib\xi}\hat{f}(a\xi)$$

Definition 2.4 *If $f \in L^1(\mathbb{R})$ and its Fourier transform $\hat{f}$ is also in $L^1(\mathbb{R})$, then the inverse Fourier transform is defined by:*

$$(\mathcal{F}^{-1}\hat{f})(x) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\hat{f}(\xi)d\xi$$

Theorem 2.5 (Fourier Inversion) *If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ then $f(x) = (\mathcal{F}^{-1}\hat{f})(x)$ at every point where f is continuous.*

The proof of the theorems mentioned above can be found in [14].

2.1.1 Fourier transforms on $L^2(\mathbb{R})$

Importantly, the Fourier transform can be extended to $L^2(\mathbb{R})$. One approach I will sketch is to firstly note that if f is in the Schwartz space $\mathcal{S}(\mathbb{R})$, then also its Fourier transform is in $\mathcal{S}(\mathbb{R})$. The inversion formula applies and we have that $(\hat{f}, \hat{g}) = 2\pi(f, g)$ for $f, g \in \mathcal{S}(\mathbb{R})$. Then thinking of $\mathcal{S}(\mathbb{R})$ as a subspace of $L^2(\mathbb{R})$, we have that $(2\pi)^{-1/2}\mathcal{F}$ is an isometry. As $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, we can extend $\mathcal{F}$ to $L^2(\mathbb{R})$. The important results we will need in subsequent chapters are the following:

Theorem 2.6 (Parseval) *If $f, g \in L^2(\mathbb{R})$, then $(f, g) = \frac{1}{2\pi}(\hat{f}, \hat{g})$*

Theorem 2.7 *The mapping $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bijection*

2.2 Fourier Series

Let $f \in L^2[0, 2\pi]$. Its Fourier series is $\sum_{k\in\mathbb{Z}} c_k e^{ikx}$, where $c_k = \left(f, e^{ikx}\right)$. It is a standard result that the sequence $\left\{(2\pi)^{-1/2} e^{ikx}\right\}$ forms an orthonormal basis of $L^2[0, 2\pi]$, hence the Fourier series of f converes to f in $L^2[0, 2\pi]$. We also have a Parseval's identity for Fourier series.

For $N \in \mathbb{Z}_{\geq 0}$, define the N^{th} partial sum of the Fourier series $S_N f$ by

$$S_N f(x) = \sum_{k=-N}^{N} c_k e^{ikx}$$

and define the N^{th} Cesàro sum $\sigma_N f = (N+1)^{-1} \sum_0^N S_i f(x)$, and the Fejér kernel

$$K_N(x) = \frac{1}{N+1} \frac{\sin^2\left(\frac{N+1}{2}x\right)}{2\sin^2\left(\frac{x}{2}\right)}$$

it can easily be shown that $\sigma_N f$ is the convolution of f with K_N. (see Chui[2] for details).

Let $\mathbb{T}$ denote the unit circle in the complex plane, then we have the following very deep result, due to Carleson[11].

Theorem 2.8 (Carleson) *If $f \in L^2(\mathbb{T})$, then the partial sums $S_N f$ converge to f almost everywhere*

Chapter 3

Multiresolution Approximations and Scaling Functions

Fundamental to wavelet theory is the notion of a multiresolution approximation of $L^2(\mathbb{R}^n)$, a concept crystallised by S. Mallat [9].

Definition 3.1 *A multiresolution approximation of $L^2(\mathbb{R}^n)$ is an increasing sequence V_j, $j \in \mathbb{Z}$ of closed subspaces of $L^2(\mathbb{R}^n)$ such that:*

1. $\bigcap\limits_{-\infty}^{\infty} V_j = \{0\}$;
2. $\bigcup\limits_{-\infty}^{\infty} V_j$ *is dense in* $L^2(\mathbb{R}^n)$;
3. $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$, *for any* $f \in L^2(\mathbb{R}^n)$ *and any* $j \in \mathbb{Z}$;
4. $f(x) \in V_0 \Longleftrightarrow f(x-k) \in V_0$, *for any* $f \in L^2(\mathbb{R}^n)$ *and any* $j \in \mathbb{Z}$; *and*
5. *there exists a function $g(x)$, known as the scaling function, such that $\{g(x-k) : k \in \mathbb{Z}^n\}$ is a Riesz basis of $L^2(\mathbb{R}^n)$.*

Because of property (2), any function in $L^2(\mathbb{R}^n)$ can be approximated arbitrarily well by its projection onto some space V_j. Moreover, as we step from a space V_j to its successor V_{j+1}, we introduce functions with double the 'frequency', as guaranteed by property (3). Roughly speaking, this means that a function f can be approximated at resolution 2^{-j} by an element of V_j.

The significance of multiresolution approximations in the wavelet theory is that

Definition 3.2 *If H is a Hilbert space, then a sequence $\{e_i\}_{i\in\mathbb{N}}$ is said to be a Riesz basis of H if its span is dense in H and if there exist constants $0 < C_1 < C_2$ such that for any sequence of scalars $\{\alpha_i\}_{i\in\mathbb{N}}$*

$$C_1\left(\sum|\alpha_k|^2\right)^{1/2} \leq \left\|\sum \alpha_k e_k\right\| \leq C_2\left(\sum|\alpha_k|^2\right)^{1/2}$$

When we get to the construction of orthonormal wavelets, we will use an orthonormal basis of V_0 which in similar in structure to the Riesz basis $\{g(x-k) : k \in \mathbb{Z}\}$. For this, the Gram-Schmidt process is not sufficient and we need a stronger result:

Theorem 3.3 *Suppose V_j is a multiresolution approximation of $L^2(\mathbb{R}^n)$. Then there exist constants $0 < c_1 \leq c_2$, such that for almost all $\xi \in \mathbb{R}^n$,*

$$c_1 \leq \left(\sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi + 2k\pi)|^2\right)^{1/2} \leq c_2$$

Proof. By definition of Riesz basis, we have that for all finite sums $\sum \alpha_k g(x-k)$,

$$C\left(\sum|\alpha_k|^2\right)^{1/2} \leq \left\|\sum \alpha_k g(x-k)\right\| \leq C'\left(\sum|\alpha_k|^2\right)^{1/2} \tag{3.1}$$

Now put $f(x) = \sum \alpha_k g(x-k)$, then $\|f\| = (f,f)^{1/2} = \left[\frac{1}{2\pi}(\hat{f},\hat{f})\right]^{1/2} = \frac{1}{\sqrt{2\pi}}\|\hat{f}\|$, by Parseval's theorem (for Fourier transforms). Take the Fourier transform: $\hat{f} = \sum \alpha_k \hat{g}(\xi)e^{-ik\xi} = \hat{g}(\xi)m(\xi)$, where, by definition, $m(\xi) = \sum \alpha_k e^{-ik\xi}$. But $m(\xi)$ is a Fourier series, so by Parseval's theorem (for Fourier series), $\|m\|_2 = \left\|\sum \alpha_k e^{-ik\xi}\right\|_2 = \sqrt{2\pi}\left(\sum|\alpha_k|^2\right)^{1/2}$, so $\left(\sum|\alpha_k|^2\right)^{1/2} = \frac{1}{\sqrt{2\pi}}\|m\|_2$. Substitute into equation 3.1, and we have:

$$C\|m\|_2 \leq \int_{\mathbb{R}^n} |m(\xi)|^2 |\hat{g}(\xi)|^2 d\xi \leq C'\|m\|_2$$

Now, observing that m is 2π-periodic, we have:

$$
\begin{aligned}
\int_{\mathbb{R}^n} |m(\xi)|^2 |\hat{g}(\xi)|^2 d\xi &= \sum_{k\in\mathbb{Z}^n} \int_{[0,2\pi)^n+2\pi k} |m(\xi)|^2 |\hat{g}(\xi)|^2 d\xi \\
&= \sum_{k\in\mathbb{Z}^n} \int_{[0,2\pi)^n} |m(\xi)|^2 |\hat{g}(\xi+2\pi k)|^2 d\xi \\
&= \int_{[0,2\pi)^n} |m(\xi)|^2 \sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi+2\pi k)|^2 d\xi
\end{aligned}
$$

and hence

$$
C \|m\|_2 \leq \left(\int_{Q_0} |m(\xi)|^2 \omega(\xi)\, d\xi \right)^{1/2} \leq C' \|m\| \tag{3.2}
$$

where, by definition, $Q_0 = [0,2\pi)^n$, and $\omega(\xi) = \sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi+2\pi k)|^2$. Now to get the result, write $m_N(\xi) = (2\pi N)^{-1/2} \sum_{0\leq k_j<N} e^{ik\cdot\xi}$, so that $|m_N(\xi)|^2$ is the Fejér kernel $K_N(\xi)$. Choose $m(\xi) = m_N(\xi-\xi_0)$, so that 3.2 becomes

$$
C \leq (K_N * \omega)^{1/2}(\xi_0) \leq C'
$$

but we know that this convolution product is merely the Cesàro sum $\sigma_N\omega$. By Carleson's theorem, the partial sums of the Fourier series converge pointwise (almost everywhere), hence the Cesàro sums must likewise converge almost everywhere to $\omega(\xi_0)$, and the result follows. ∎

Using this, we can construct an orthonormal basis as follows:

Theorem 3.4 *Let $\phi \in L^2(\mathbb{R}^n)$ such that*

$$
\hat{\phi}(\xi) = \hat{g}(\xi) \left(\sum_{k\in\mathbb{Z}^n} |\hat{g}(\xi+2k\pi)|^2 \right)^{-1/2}
$$

then $\{\phi(x-k) : k\in\mathbb{Z}^n\}$ is an orthonormal basis of V_0.

Proof. Indeed, we will show that $\{\phi(x-k) : k\in\mathbb{Z}^n\}$ is the pre-image of an orthonormal basis of $L^2\left(Q_0, |Q_0|^{-1} d\xi\right)$ under an isometric isomorphism.

So consider the map $U : V_0 \to L^2\left(Q_0, |Q_0|^{-1} d\xi\right)$, defined by $\sum \alpha_k g(x-k) \longmapsto m(\xi)(\omega(\xi))^{1/2}$ (employing the notation of the previous theorem, so that $Uf(\xi).\hat{g}(\xi) = \hat{f}(\xi)(\omega(\xi))^{1/2}$). We claim this is an isometric isomorphism.

As the finite sums $\sum \alpha_k g(x-k)$ are dense in V_0, we first show that U preserves the norm of finite sums, then the result will hold in all of V_0 by continuity. But, writing $f = \sum \alpha_k g(x-k)$,

$$\|f\|_2 = |Q_0|^{-1/2} \left(\int_{Q_0} |m(\xi)|^2 \omega(\xi)\, d\xi \right)^{1/2}$$

using the same arguments as in the previous theorem.

Now let $h \in L^2\left(Q_0, |Q_0|^{-1} d\xi\right)$. Because we have $0 < c_1 \leq \omega(\xi)^{1/2} \leq c_2$, we also have $h/\left(\omega^{1/2}\right) \in L^2\left(Q_0, |Q_0|^{-1} d\xi\right)$, so there exists a sequence $\{\alpha_k : k \in \mathbb{Z}^n\}$ with $\sum |\alpha_k|^2 < \infty$, and $h/\left(\omega^{1/2}\right) = \sum \alpha_k e^{-ik\cdot\xi} =: m(\xi)$, then $h(\cdot) = m(\cdot)(\omega(\cdot))^{1/2} \in \operatorname{Im} U$, hence U is surjective.

Note that for any $\nu \in V_0$, $U(\nu(\cdot - k)) = e^{-ik\cdot\xi} U\nu$, so we seek an orthonormal basis of $L^2\left(Q_0, |Q_0|^{-1} d\xi\right)$ of the form $\left\{e^{-ik\cdot\xi} U(\phi) : k \in \mathbb{Z}^n\right\}$. Choose ϕ so that $U(\phi) = 1$, whence $\hat{\phi}(\xi) = U\phi(\xi) . \hat{g}(\xi)(\omega(\xi))^{-1/2} = \hat{g}(\xi)(\omega(\xi))^{-1/2}$, as required. ∎

3.1 Regular multiresolution approximations

Definition 3.5 *Let V_j be a multiresolution approximation of $\mathbb{R}^n$. It is said to be r-regular, for a natural number r, if the function g (which generates a Riesz basis of V_0) satisfies the property that for any sequence $\alpha_1, \ldots, \alpha_n$ of non-negative integers such that $\sum \alpha_j \leq r$, and for any natural number m, there exists a constant C_m such that for any $x \in \mathbb{R}^n$,*

$$\left|(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n} g(x)\right| \leq C_m (1+|x|)^{-m}$$

There are many examples of regular multiresolution approximations, and indeed, our next example shows a construction of an r-regular multiresolution approximation for each $r \in \mathbb{N}$.

Example 3.6 (splines of order r) *As is typical in dealing with multiresolution approximations, we first define the space V_0, and then use property 3 to define the rest of the spaces. The functions in V_0 are the ($L^2(\mathbb{R})$) functions $f \in C^{r-1}$ such that for each $k \in \mathbb{Z}$, the restriction of f to $[k, k+1)$ is a polynomial of degree $\leq r$. Define χ to be the characteristic function of* $[0,1]$, *and define*

$$g = \underbrace{\chi * \ldots * \chi}_{r+1}$$

Then $\{g(x-k) : k \in \mathbb{Z}\}$ *is a Riesz basis of* V_0. *For instance, in the case* $r = 1$ *we have*

$$\begin{aligned} g(x) &= \chi * \chi(x) \\ &= \int_{\mathbb{R}} \chi(x-y)\,\chi(y)\,dy \\ &= \int_{\mathbb{R}} \chi_{[-1+x,x]}(y)\,\chi_{[0,1]}(y)\,dy \\ &= |[-1+x,x] \cap [0,1]| \end{aligned}$$

this is a symmetric tent-like function with base $[0,2]$ *and unit height. It is clear that the span of these tent-like functions is* V_0, *also the function is* 1*-regular (first derivative vanishes as you move away from the origin).*

Importantly, if we have a regular multiresolution approximation, we can deduce that the scaling function ϕ is also in some sense regular, as demonstrated by the following theorem, which is stated without proof (see Meyer[1]).

Lemma 3.7 *If for every* $m \in \mathbb{N}$, *there exists a constant* C_m, *such that* $|g(x)| \le C_m (1+|x|)^{-m}$, *then* $\left(\sum |\hat{g}(\xi + 2k\pi)|^2\right)^{1/2} \in C^\infty$.

Theorem 3.8 *Suppose* V_j *is an* r*-regular multiresolution approximation of* $L^2(\mathbb{R}^n)$. *Then the scaling function* ϕ *as defined in 3.4 has the property that for any sequence* $\alpha_1, \ldots, \alpha_n$ *of non-negative integers such that* $\sum \alpha_j \le r$, *and for any natural number* m, *there exists a constant* C_m *such that for any* $x \in \mathbb{R}^n$,

$$|(\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}\, \phi(x)| \le C_m (1+|x|)^{-m}$$

Chapter 4

Orthonormal wavelets

After having discussed the notion of the multiresolution approximation, we are now in a position to look at the central objects under consideration, namely the orthonormal wavelet bases of $L^2(\mathbb{R}^n)$. Before I give the definition, I will present some arguments to motivate the concept.

In Fourier analysis, we have an orthonormal basis of $L^2(0, 2\pi)$ which is generated from a single function e^{ix} by taking higher frequency versions. We can't use the e^{inx} as a basis for $L^2(\mathbb{R})$ as the e^{inx} aren't even in $L^2(\mathbb{R})$. An aspirant basic wavelet ψ must decay rapidly at $\pm\infty$, hence must be sufficiently localised. Because of this localisation, we are forced to consider translates of the function $\psi(x-k)$, $k \in \mathbb{Z}$, if we wish to cover all of $\mathbb{R}$. This is still not enough because we would like to be able to use wavelets to approximate an $L^2(\mathbb{R})$ function f up to any given resolution, so we need to loot at binary dilations $\psi\left(2^j x - k\right)$, finally we need to introduce a factor of $2^{j/2}$ to ensure each wavelet has unit norm (assuming the basic wavelet ψ has unit norm).

4.1 One-dimensional wavelets

Definition 4.1 *Let $m \in \mathbb{N}$, and let $\psi \in L^\infty(\mathbb{R})$. ψ is said to be a basic wavelet of class m if:*

1. *for $0 \le i \le m$, $\psi^{(i)} \in L^\infty(\mathbb{R})$;*
2. *for every $m \ge 1$ and for $0 \le i \le m$, there exists a constant C_m, such that for every $x \in \mathbb{R}$, $\left|\psi^{(i)}(x)\right| \le C_m(1+|x|)^{-m}$;*

3. *for* $0 \le k \le m$, $\int_{-\infty}^{\infty} x^k \psi(x)dx = 0$;

4. $\{2^{j/2}\psi(2^j x - k) : j, k \in \mathbb{Z}\}$ *is an orthonormal basis of* $L^2(\mathbb{R})$

Definition 4.2 *Let* ψ *be a basic (or mother) wavelet, as defined above, then the functions* $2^{j/2}\psi(2^j x - k)$ *where* $j, k \in \mathbb{Z}$ *are called wavelets.*

The wavelets are intimately connected with the set of dyadic intervals,

$$\mathcal{I} = \left\{[k2^{-j}, (k+1)2^{-j}) : j, k \in \mathbb{Z}\right\}$$

Indeed, let $I = [k2^{-j}, (k+1)2^{-j}) \in \mathcal{I}$, and define $\psi_{j,k} = \psi_I = 2^{j/2}\psi(2^j \cdot - k)$, then ψ_I is localised in the interval I, and for a fixed $j \in \mathbb{Z}$, the wavelets $\left\{\psi_I : |I| = 2^{-j}\right\}$ serve to resolve an L^2 function f at a 'granularity' of 2^{-j}.

The obvious question is how do we construct these basic wavelets. The route I will use is to introduce a so-called scaling function ϕ. The idea is that we will use the ϕ to resolve f at granularities greater that 1, and then use wavelets of smaller granularity for the finer details. Specifically, we will require that the union of the $\phi(\cdot - k), k \in \mathbb{Z}$, and the $\psi_I, |I| \le 1$, will form an orthonormal basis of $L^2(\mathbb{R})$, so that for $f \in L^2(\mathbb{R})$,

$$f(x) = \sum_{k=-\infty}^{\infty} (f, \phi(\cdot - k))\, \phi(x-k) + \sum_{I \in \mathcal{I}, |I| \le 1} (f, \psi_I)\, \psi_I \tag{4.1}$$

The reason for introducing scaling functions is that we can now find a relation between multiresolution approximations and wavelets. The first observation is that the basic wavelet ψ and the scaling function ϕ give rise to a multiresolution approximation of $L^2(\mathbb{R})$ as follows:

Define $\phi_{j,k}(x) = 2^{j/2}\phi\left(2^j x - k\right)$, fix $j \in \mathbb{Z}$, then

$$\begin{aligned} \left(\phi_{j,k}, \phi_{j,k'}\right) &= \int [2^{j/2}\phi\left(2^j x - k\right)][2^{j/2}\phi\left(2^j x - k'\right)]dx \\ &= 2^j \int \phi(s-k)\, \phi\left(s-k'\right) 2^{-j} ds \\ &= \left(\phi_{1,k}, \phi_{1,k'}\right) \end{aligned}$$

but by hypothesis, the sequence $\left\{\phi_{1,k}\right\}_{k \in \mathbb{Z}}$ is orthonormal, hence so is the sequence $\left\{\phi_{j,k}\right\}_{k \in \mathbb{Z}}$

for any $j \in \mathbb{Z}$.

Define $V_j = \overline{span\left\{\phi_{j,k}\right\}_{k\in\mathbb{Z}}}$, so that $\left\{\phi_{j,k}\right\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of V_j. Note that we can transform equation 4.1, using the substitution $x = 2^j s$ to give:

$$f(s) = \sum_{k=-\infty}^{\infty} 2^j (f, \phi_{j,k}) \phi_{j,k}(s) + \sum_{I\in\mathcal{I},|I|\leq 2^{-j}} (f, \psi_I)\, \psi_I(s)$$

Let W_j be the orthogonal complement of V_j in V_{j+1}, so that $V_{j+1} = V_j \oplus W_j$. By the last relation, we find that the wavelets $\left\{\psi_{j,k}\right\}_{k\in\mathbb{Z}}$ form an orthonormal basis of W_j, we also see that $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$. The other properties of a multiresolution analysis are easily verified, in particular, the function ϕ generates an orthonormal (and hence a Riesz) basis of the space V_0.

4.1.1 Construction of one-dimensional wavelet bases

Happily, we can also go the other way as well. That is, given an r-regular multiresolution approximation V_j of $L^2(\mathbb{R})$, we can find a basic wavelet of class r, generated by the approximation.

Before we begin the construction, it is interesting to note that it is not possible to have an 'infinitely regular' compactly supported basic wavelet because of a theorem of Hausdorff, which states that if f and g are functions continuous on a closed interval and all their k^{th} moments are equal, then $f = g$ on the interval (see Körner[12] for a proof).

Let V_j be an r-regular multiresolution approximation of $L^2(\mathbb{R})$, define W_j to be the orthogonal complement of V_j in V_{j+1}. We will start by constructing a function $2^{-1/2}\psi(x/2) \in W_{-1}$, which is our putative basis generator. Recall that we have a scaling function $\phi \in V_0$ which satisfies the properties of theorem 3.4 and theorem 3.8.

Lemma 4.3 *There exists a unitary mapping U from $\mathcal{F}V_0$ to $L^2[0, 2\pi]$.*

Proof. Let $f \in V_0$, so we may write $f(x) = \sum \alpha_k \phi(x-k)$ for some sequence of scalars α_k, then taking the Fourier transform,

$$\begin{aligned} \hat{f}(\xi) &= \sum \alpha_k e^{-ikx} \hat{\phi}(\xi) \\ &= \hat{\phi}(\xi)\, m(\xi) \end{aligned}$$

where we have defined $m(\xi) = \sum \alpha_k e^{-ikx}$. We have that $m(\xi)$ is 2π-periodic and as $\sum |\alpha_k|^2 < \infty$, we also have that $m \in L^2[0, 2\pi]$. Define $U : \mathcal{F}V_0 \to L^2[0, 2\pi]$ by $U\left(m(\xi)\,\hat{\phi}(\xi)\right) = m(\xi)$. U is clearly a linear bijection, and it's norm-preserving as $\left(\sum |\alpha_k|^2\right)^{1/2} = \|f\|_2$, but

$$m(\xi) = \sum \alpha_k e^{-ik\xi} = \sqrt{2\pi} \sum \alpha_k \frac{e^{-ik\xi}}{\sqrt{2\pi}}$$

and using the fact that $\left\{\frac{e^{-ik\xi}}{\sqrt{2\pi}}\right\}$ is an orthonormal basis of $L^2[0, 2\pi]$, we have

$$\left\| \frac{m}{\sqrt{2\pi}} \right\|_2 = \left\| \sum \alpha_k \frac{e^{-ik\xi}}{\sqrt{2\pi}} \right\|_2 = \left(\sum |\alpha_k|^2\right)^{1/2}$$

hence $\|f\|_2 = \frac{1}{\sqrt{2\pi}} \|m\|_2$, and by Parseval's theorem, $\left\|\hat{f}\right\| = \sqrt{2\pi}\,\|f\|_2 = \|m\|_2$. ■

Lemma 4.4 $\sum_{l \in \mathbb{Z}} \left|\hat{\phi}(\xi + 2\pi l)\right|^2 = 1$ *almost everywhere.*

Proof. By orthonormality of the sequence $\{\phi(x-k) : k \in \mathbb{Z}\}$, we have

$$\begin{aligned}
\delta_{k,0} &= \int_{\mathbb{R}} \phi(x-k)\overline{\phi(x)}dx \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ik\xi} \phi(\xi)\,\overline{\phi(\xi)} d\xi \\
&= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\xi} \sum_{l \in \mathbb{Z}} \left|\hat{\phi}(\xi + 2\pi l)\right|^2 d\xi
\end{aligned}$$

put $\nu(\xi) = \sum_{l \in \mathbb{Z}} \left|\hat{\phi}(\xi + 2\pi l)\right|^2$, we've shown the k^{th} Fourier coefficient of $\nu(\xi)$ is $\delta_{k,0}$. Therefore the $\nu(\xi) = 1$ almost everywhere ■

Now $\phi(x/2) \in V_{-1} \subset V_0$, so there exists a sequence α_k of scalars such that

$$\frac{1}{2}\phi\left(\frac{x}{2}\right) = \sum \alpha_k \phi(x + k)$$

taking the Fourier transform

$$\widehat{\frac{1}{2}\phi\left(\frac{x}{2}\right)}(\xi) = \hat{\phi}(2\xi) = m_0(\xi)\,\hat{\phi}(\xi)$$

where $m_0(\xi) = \sum \alpha_k e^{ik\xi}$. By lemma 4.4, we have $\sum \left|\hat{\phi}(\xi + 2\pi k)\right|^2 = 1$, hence $\sum \left|\hat{\phi}(2\xi + 2\pi k)\right|^2 =$

1, and substituting for $\hat{\phi}(2\xi)$, $\sum |m_0(\xi + k\pi)|^2 \left|\hat{\phi}(\xi + k\pi)\right|^2 = 1$, but m_0 is 2π-periodic, so

$$\begin{aligned} 1 &= |m_0(\xi+\pi)|^2 \sum_{k \text{ odd}} \left|\hat{\phi}(\xi + k\pi)\right|^2 + |m_0(\xi)|^2 \sum_{k \text{ even}} \left|\hat{\phi}(\xi + k\pi)\right|^2 \qquad (4.2) \\ &= |m_0(\xi+\pi)|^2 + |m_0(\xi)|^2 \left(\sum_{k\in\mathbb{Z}} \left|\hat{\phi}(\xi + k\pi)\right|^2 \right) \\ &= |m_0(\xi+\pi)|^2 + |m_0(\xi)|^2 \end{aligned}$$

as in a previous lemma, we have that

$$\begin{aligned} \mathcal{F}V_{-1} &= \left\{ m(2\xi)\,\hat{\phi}(2\xi) : m \in L^2[0,2\pi] \right\} \\ &= \left\{ m(2\xi)\, m_0(\xi)\,\hat{\phi}(\xi) : m \in L^2[0,2\pi] \right\} \end{aligned}$$

so $U\mathcal{F}V_{-1} = \left\{ m(2\xi)\, m_0(\xi) : m \in L^2[0,2\pi] \right\}$

Our ultimate aim is to characterise the space W_{-1}, which is the orthogonal complement of V_{-1} in V_0. We will do this by looking at the orthogonal complement of $U\mathcal{F}V_{-1}$ in $U\mathcal{F}V_0 = L^2[0,2\pi]$. Suppose m_f belongs to the complement $U\mathcal{F}W_{-1}$, then for each $m \in L^2[0,2\pi]$,

$$\begin{aligned} 0 &= \int_0^{2\pi} m(2\xi)\, m_0(\xi)\, \overline{m_f(\xi)} d\xi \\ &= \int_0^{\pi} m'(\xi) \left[m_0(\xi)\, \overline{m_f(\xi)} + m_0(\xi+\pi)\, \overline{m_f(\xi+\pi)} \right] d\xi \end{aligned}$$

where $m'(\xi) = m(2\xi)$ is any π-periodic function in $L^2[0,\pi]$, so we have, equivalently,

$$m_0(\xi)\, \overline{m_f(\xi)} + m_0(\xi+\pi)\, \overline{m_f(\xi+\pi)} = 0$$

by 4.2, we have that for fixed ξ, the vector $(m_0(\xi), m_0(\xi+\pi))$ is a unit vector in $\mathbb{C}^2$, and the above relation states that the vector $(m_f(\xi), m_f(\xi+\pi))$ is orthogonal to $(m_0(\xi), m_0(\xi+\pi))$. But also $\left(\overline{m_0(\xi+\pi)}, -\overline{m_0(\xi)}\right)$ is orthogonal to $(m_0(\xi), m_0(\xi+\pi))$, and because $\mathbb{C}^2$ is two-dimensional, there exists $\beta(\xi)$ such that

$$(m_f(\xi), m_f(\xi+\pi)) = \beta(\xi)\, e^{-i\xi} \left(\overline{m_0(\xi+\pi)}, -\overline{m_0(\xi)} \right)$$

indeed,

$$\begin{aligned}
\beta(\xi) &= \left\langle (m_f(\xi), m_f(\xi+\pi)), e^{-i\xi}\left(\overline{m_0(\xi+\pi)}, -\overline{m_0(\xi)}\right)\right\rangle \\
&= e^{i\xi}[m_f(\xi)\, m_0(\xi+\pi) - m_f(\xi+\pi)\, m_0(\xi)]
\end{aligned}$$

So we have that $\beta(\xi)$ is π-periodic and $m_f(\xi) = \beta(\xi)\, e^{-i\xi}\overline{m_0(\xi+\pi)}$. This defines a mapping F from $U\mathcal{F}W_{-1}$ to $L^2[0,\pi]$, given by $F : m_f \mapsto \beta$. The mapping can be seen to be an isometric isomorphism, so we have an orthonormal basis of $U\mathcal{F}W_{-1}$ given by

$$\left\{e^{-i\xi}\overline{m_0(\xi+\pi)}e^{2ki\xi} : k \in \mathbb{Z}\right\}$$

(this being the preimage under F of the orthonormal basis $e^{2ki\xi}$ of $L^2[0,\pi]$). Inverting U, we have an orthonormal basis of $\mathcal{F}W_{-1}$ given by $\left\{e^{-i\xi}\overline{m_0(\xi+\pi)}e^{2ki\xi}\hat{\phi}(\xi) : k \in \mathbb{Z}\right\}$. Let ψ be a function such that

$$\hat{\psi}(2\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)$$

then

$$\begin{aligned}
2^{-1/2}\widehat{\psi(2^{-1}x-k)}(\xi) &= 2^{-1/2}\widehat{\psi\left(\frac{x-2k}{2}\right)}(\xi) \\
&= 2^{-1/2}.2.e^{-2ki\xi}\hat{\psi}(2\xi) \\
&= 2^{-1/2}e^{-2ki\xi}e^{-i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)
\end{aligned}$$

and hence $\left\{2^{-1/2}\psi\left(2^{-1}x-k\right) : k \in \mathbb{Z}\right\}$ is an orthonormal basis of W_{-1}.

We go from W_{-1} to W_0 by the following remark. Suppose that $f \in W_j \subset V_{j+1}$, then by property 3 of multiresolution approximations, we have $f(2\cdot) \in V_{j+2}$, so

$$\overbrace{f}^{\in V_{j+1}} = \overbrace{0}^{\in V_j} + \overbrace{f}^{\in W_j}$$

but using the fact that $V_{j+2} = V_{j+1} \oplus W_{j+1}$,

$$\overbrace{f(2\cdot)}^{\in V_{j+2}} = \overbrace{0}^{\in V_{j+1}} + \overbrace{f(2\cdot)}^{\in W_{j+1}}$$

we have $f \in W_j \Longrightarrow f(2\cdot) \in W_{j+1}$, and going the other way, $f(2\cdot) \in W_{j+1} \Longrightarrow f \in W$.

So we have that $\{\psi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis of W_0. By construction, ψ has the same localisation properties as ϕ, that is, for every $m \geq 1$ and for $0 \leq i \leq m$, there exists a constant C_m, such that for every $x \in \mathbb{R}$, $\left|\psi^{(i)}(x)\right| \leq C_m(1+|x|)^{-m}$. To get the oscillatory property of wavelets, we use the following propostition:

Proposition 4.5 $\left|\hat{\phi}(\xi)\right|^2 = 1 + O\left(|\xi|^{2r+2}\right)$ *as* $|\xi|$ *tends to 0.*

The proof of the proposition involves pseudo-differential operators and is omitted here. The interested reader can find a proof in Meyer[1].

We also claim that $\left|\hat{\phi}(\xi)\right|^2 + \left|\hat{\psi}(\xi)\right|^2 = \left|\hat{\phi}(\xi/2)\right|^2$, equivalently, one has to show $\left|\hat{\phi}(2\xi)\right|^2 + \left|\hat{\psi}(2\xi)\right|^2 = \left|\hat{\phi}(\xi)\right|^2$. We rewrite the left-hand side to give

$$\left|m_0(\xi)\,\hat{\phi}(\xi)\right|^2 + \left|e^{-i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)\right|^2 = \left|\hat{\phi}(\xi)\right|^2$$

but this is true, by 4.2. Then using the proposition, we have $\hat{\psi}(\xi) = O\left(|\xi|^{r+1}\right)$. Because ψ decreases rapidly at $\pm\infty$, we have that all its moments are absolutely integrable, so the Fourier transform $\hat{\psi}$ of ψ is infinitely differentiable (see [14]) and so $\widehat{(-ix)^k\psi}(\xi) = \hat{\psi}^{(k)}(\xi) = O\left(|\xi|^{r+1-k}\right)$ and so we have proved that

$$\int x^k \psi(x)\,dx = 0 \qquad \text{if } 0 \leq k \leq r$$

Using the above remark, we have that $\left\{2^{j/2}\psi(2^j x - k) : k \in \mathbb{Z}\right\}$ is an orthonormal basis of W_j. We then have $V_j = \bigoplus_{k=-\infty}^{j-1} W_k$, so that by property 2 of multiresolution approximations, $L^2(\mathbb{R}) = \bigoplus_{-\infty}^{\infty} W_k$, and so the sequence $\left\{2^{j/2}\psi\left(2^j x - k\right) : j,k \in \mathbb{Z}\right\}$ forms an orthonormal basis of $L^2(\mathbb{R})$. So we've finally constructed an r-regular orthonormal wavelet basis of $L^2(\mathbb{R})$ from an r-regular multiresolution approximation.

4.1.2 Wavelets of compact support

There are a whole bunch of multiresolution approximations to choose from, but unfortunately, the easy examples don't typically give rise to wavelet bases of compact support. We need compactly supported wavelets because they can be used in the analysis of spaces other that $L^2(\mathbb{R})$, in particular, we'll use them in looking at the real Hardy space $H^1(\mathbb{R}^n)$.

What we will do is to find an r-regular mutliresolution approximation of $L^2(\mathbb{R})$ which will give rise to a scaling function ϕ and a basic wavelet ψ, as constructed in section 4.1.1, so that ψ and ϕ are compactly supported. The construction will start from a trigonometric polynomial $m_0(\xi) = \sum_{|k|\leq T} \alpha_k e^{ik\xi}$. By analogy with the construction in the previous section, we will try to get an orthonormal sequence $\{\phi(x-k) : k \in \mathbb{Z}\}$ by defining ϕ such that

$$\hat{\phi}(2\xi) = m_0(\xi)\,\hat{\phi}(\xi) \tag{4.3}$$

This will define the closed subspace V_0 for which $\{\phi(x-k) : k \in \mathbb{Z}\}$ is an orthonormal basis, and we'll see that this leads to a multiresolution approximation.

Now if ϕ were a scaling function for some multiresolution approximation and m_0 was the 2π-periodic function as defined in section 4.1.1, and m_0 is a trigonometric polynomial, we certainly have that 4.3 holds so that for a fixed integer T,

$$\begin{aligned}
\hat{\phi}(\xi) &= m_0(\xi/2)\,\hat{\phi}(\xi/2) \\
&= m_0(\xi/2)\,m_0(\xi/4)\hat{\phi}(\xi/4) \\
&\vdots \\
&= \hat{\phi}\left(2^{-T}\xi\right)\prod_{j=1}^{T} m_0\left(2^{-j}\xi\right)
\end{aligned}$$

We claim that the infinite product converges locally uniformly (that is, it converges uniformly on compact sets). But by proposition 4.5, we have $\hat{\phi}(\xi) = 1 + O\left(|\xi|^{2r+2}\right)$, and hence $m_0(0) = 1$ (again using 4.3). So we know that the terms are eventually non-zero. Now

$$\begin{aligned}
|m_0(\xi)| &\leq 1 + |m_0(\xi) - 1| \\
&= 1 + \left|\sum_k \alpha_k e^{ik\xi} - 1\right| \\
&= 1 + \left|\sum_k \beta_k e^{ik\xi}\right|
\end{aligned}$$

(where $\beta_k = \alpha_k$ if $k \neq 0$ and $\beta_k = \alpha_k - 1$ for $k = 0$.)

$$\begin{aligned} &= 1 + \left| \sum_k \beta_k \left(\cos(k\xi) + i \sin(k\xi) \right) \right| \\ &\leq 1 + \sum_k |\beta_k| \left| \sin(k\xi) \right| \\ &\leq 1 + N \sup |\beta_k| \, |N\xi| \end{aligned}$$

(where N is the number of non-zero β_ks. Put $C = N^2 \sup |\beta_k|$)

$$\begin{aligned} &= 1 + C|\xi| \\ &\leq e^{C|\xi|} \end{aligned}$$

So we arrive at:

$$\prod_{j=1}^{\infty} \left| m_0 \left(2^{-j}\xi \right) \right| \leq \exp\left(\sum_{j=1}^{\infty} \left| C 2^{-j} \xi \right| \right) \leq e^{C|\xi|}$$

hence we have locally uniform convergence.

Of course, we want to go the other way. The first result in that direction is the following:

Lemma 4.6 (Mallat) *If m_0 is a 2π-periodic function such that $|m_0(\xi)|^2 + |m_0(\xi + \pi)|^2 = 1$ and if $\prod_{j=1}^{\infty} m_0 \left(2^{-j}\xi \right)$ converges pointwise almost everywhere, then its limit $\hat{\phi}(\xi)$ is in $L^2(\mathbb{R})$, and $\|\phi\|_{L^2} \leq 1$.*

Proof. Put $f_k(\xi) = [\prod_{j=1}^{k} m_0 \left(2^{-j}\xi \right)] \chi_{[-\pi,\pi]} \left(2^{-k}\xi \right)$, so that f_k converges to $\hat{\phi}$ almost everywhere. We also have

$$\begin{aligned} \|f_k\|^2 = \int |f_k(\xi)|^2 \, d\xi &= \int_{-2^k\pi}^{2^k\pi} \prod_{j=1}^{k} \left| m_0 \left(2^{-j}\xi \right) \right|^2 d\xi \\ &= \int_0^{2^{k+1}\pi} \prod_{j=1}^{k} \left| m_0 \left(2^{-j}\xi \right) \right|^2 d\xi \end{aligned}$$

where we have made use of the observation that each $m_0 \left(2^{-j}\xi \right)$ is $2^{j+1}\pi$-periodic and so the integrand is $2^{k+1}\pi$-periodic. Instead of integrating $\int_0^{2^{k+1}\pi}$, we can integrate $\int_0^{2^k\pi} + \int_{2^k\pi}^{2^{k+1}\pi}$, then use a substitution $u = \xi - 2^k\pi$ on the second integral to get

$$
\begin{aligned}
&= \int_0^{2^k\pi} [\prod_{j=1}^{k} \left|m_0\left(2^{-j}\xi\right)\right|^2] + [\prod_{j=1}^{k} \left|m_0\left(2^{-j}(\xi - 2^k\pi)\right)\right|^2] d\xi \\
&= \int_0^{2^k\pi} [\prod_{j=1}^{k-1} \left|m_0\left(2^{-j}\xi\right)\right|^2][\left|m_0\left(2^{-k}\xi\right)\right|^2 + \left|m_0\left(2^{-k}\xi + \pi\right)\right|^2] d\xi \\
&= \int_0^{2^k\pi} \prod_{j=1}^{k-1} \left|m_0\left(2^{-j}\xi\right)\right|^2 d\xi \\
&= \|f_{k-1}\|^2
\end{aligned}
$$

So we have

$$
\begin{aligned}
\|f_k\|^2 = \|f_{k-1}\|^2 = \ldots &= \|f_1\|^2 \\
&= \int_{-2\pi}^{2\pi} \left|m_0\left(\xi/2\right)\right|^2 d\xi \\
&= 2\int_{-\pi}^{\pi} \left|m_0\left(\xi\right)\right|^2 d\xi = 2\pi
\end{aligned}
$$

using $\left|m_0\left(\xi\right)\right|^2 + \left|m_0\left(\xi + \pi\right)\right|^2 = 1$.

Apply Fatou's lemma,

$$
\int \left|\hat{\phi}\left(\xi\right)\right|^2 d\xi \leq \limsup_{k\to\infty} \int \left|f_k\left(\xi\right)\right|^2 \leq 2\pi
$$

and then Parseval gives the desired inequality. ■

Before we use Mallat's lemma to show that the $\{\phi\left(x - k\right) : k \in \mathbb{Z}\}$ are orthonormal, we will first demonstrate that ϕ is compactly supported. Before this we need the Paley-Wiener theorem, which I state without proof:

Theorem 4.7 (Paley-Wiener) *An entire function $\hat{T}\left(\xi\right)$ is the analytic extenstion of the Fourier transform of a distribution T in $\mathcal{S}'\left(\mathbb{R}\right)$ supported in $[a, b]$ iff for some $N \in \mathbb{N}$, and $C_N > 0$*

$$
\left|\hat{T}\left(\xi\right)\right| \leq C_N\left(1 + |\xi|\right)^N \begin{cases} e^{b\,\mathrm{Im}\,\xi} & \mathrm{Im}\,\xi \geq 0 \\ e^{a\,\mathrm{Im}\,\xi} & \mathrm{Im}\,\xi \leq 0 \end{cases}
$$

Lemma 4.8 *Let m_0 be a trigonometric polynomial, say there exists $T \in \mathbb{N}$ and scalars α_k such* *that*

$$m_0 = \sum_{|k| \leq T} \alpha_k e^{ik\xi}$$

and that $m_0(0) = 1$. *Define* $\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m_0\left(2^{-j}\xi\right)$, *then* ϕ *is compactly supported*

Proof. We can use the argument I presented at the beginning of this section to show that the partial products converge locally uniformly to $\hat{\phi}$, since each of the functions $m_0\left(2^{-j}\xi\right)$ is entire, we have that $\hat{\phi}$ is (see [15]). Again using the previous arguments, we have

$$\left|\hat{\phi}(\xi)\right| \leq e^{C|\xi|}$$

One can derive other inequalities to get both of the polynomial bounds as required (see Daubechies[8], for example), and we have that ϕ is compactly supported. ■

Finally we have:

Lemma 4.9 *If* $m_0(\xi)$ *is a trigonometric polynomial and* $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$ *and* $m_0(0) = 1$ *and* $m_0(\xi) \neq 0$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$, *then* $\{\phi(x-k) : k \in \mathbb{Z}\}$ *is an orthonormal sequence.*

Proof. Put $\alpha(\xi) = \sum_{-\infty}^{\infty} \left|\hat{\phi}(\xi + 2k\pi)\right|^2$, then working lemma 4.4 backwards, we only have to show that $\alpha \equiv 1$. This is a consequence of the identity $\alpha(2\xi) = \alpha(\xi) g(\xi) + \alpha(\xi+\pi) g(\xi+\pi)$. See Meyer[1] for details. ■

Why is ψ compactly supported? Well $\hat{\psi}(2\xi) = e^{-i\xi}\overline{m_0(\xi+\pi)}\hat{\phi}(\xi)$ and a Paley-Wiener argument gives compact support. The tricky parts which are left to show are that the V_j do indeed form a multiresolution approximation of $L^2(\mathbb{R}^n)$, and that the mutliresolution approximation is r-regular. Again, see either Meyer [1] or Daubechies[8] for the proofs.

4.2 Multi-dimensional wavelets

There are two ways to proceed in the construction of multidimensional wavelets. One approach is to start with a multiresolution approximation of $L^2(\mathbb{R}^n)$, and then to generalise the construction of the previous section. The problem with this method is that it won't in general lead to compactly supported wavelets, even if the scaling function ϕ is compactly supported. Consequently, I will use the alternative, 'tensor product' approach in which we construct a higher dimensional multiresolution approximation as the algebraic tensor product of one-dimensional multiresolution approximations

Specifically, suppose that $\mathcal{V}_j$ is a multiresolution approximation of $L^2(\mathbb{R})$. We define $V_j \subset L^2(\mathbb{R}^n)$ by $V_j = \overline{\bigotimes_{i=1}^n \mathcal{V}_j}$. Then by definition, we have that

$$\{\phi(x_1 - k_1)\ldots\phi(x_2 - k_n) : (k_1, \ldots, k_n) \in \mathbb{Z}^n\}$$

is an orthonormal basis of V_0. It is easily seen that V_j is a multiresolution approximation of $L^2(\mathbb{R}^n)$. We let W_j denote the orthogonal complement of V_j in V_{j+1}, and analogously to the one-dimensional case, we will seek a collection of basic wavelets ψ^ε (a single basic wavelet isn't sufficient in the multidimensional case) which generate an orthonormal basis of W_0. Now let $\mathcal{W}_j$ denote the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$. We have

$$\begin{aligned}
V_1 &= \overline{\bigotimes_{i=1}^n \mathcal{V}_1} \\
&= \overline{\bigotimes_{i=1}^n (\mathcal{V}_0 \oplus \mathcal{W}_0)} \\
&= \overline{\bigotimes_{i=1}^n \mathcal{V}_0} \oplus \bigoplus_{\varepsilon \in E} W^\varepsilon \\
&= V_0 \oplus \bigoplus_{\varepsilon \in E} W^\varepsilon
\end{aligned}$$

where we define $E = \{(\varepsilon_1, \ldots, \varepsilon_n) : \varepsilon_j \in \{0, 1\}\} \setminus \{(0, \ldots, 0)\}$ and

$$W^\varepsilon = \overline{\bigotimes_{i=1}^n A_i}$$

with $A_i = V_0$ if $i = 0$ and $A_i = W_0$ if $i = 1$. Consequently, we have that $W_0 = \bigoplus\limits_{\varepsilon \in E} W^\varepsilon$, and each W^ε has an orthonormal basis which we get from multiplying bases of $\mathcal{V}_0$ and $\mathcal{W}_0$, that is, if we define $\psi^0 = \phi$ and $\psi^1 = \psi$, then W^ε has basis

$$\{\psi^\varepsilon(x - k) : k \in \mathbb{Z}^n\} := \left\{\prod_{i=1}^n \psi^{\varepsilon_i}(x_i - k_i) : (k_1, \ldots, k_n) \in \mathbb{Z}^n\right\}$$

and so we define the collection $\{\psi^{\varepsilon} : \varepsilon \in E\}$ to be the set of basic wavelets. Correspondingly, if we define $\mathcal{Q}$ to be the set of dyadic cubes as in 6.2, then the wavelets are given by

$$\psi_Q^{\varepsilon}(x) = 2^{nj/2}\psi^{\varepsilon_1}\left(2^j x_1 - k_1\right)\ldots\psi^{\varepsilon_n}\left(2^j x_n - k_n\right)$$

we see that these wavelets are compactly supported (assuming that ϕ and ψ are compactly supported). Indeed suppose that the supports of ϕ and ψ are contained in the interval $\left[\frac{1}{2} - \frac{m}{2}, \frac{1}{2} + \frac{m}{2}\right]$ for some $m \geq 1$. Now each function $\psi^{\varepsilon_i}\left(2^j x_i - k_i\right)$ is formed from ψ^{ε_i} by stretching by 2^{-j} and then moving it over to $k_i 2^{-j}$, so is supported in an interval centered around $k_i 2^{-j}$, of diameter m, and hence the product ψ_Q^{ε} is supported in the cube mQ. One can verify that the wavelets constructed in this manner satisfy the properties which one expects from wavelets (i.e. that they form a basis of $L^2(\mathbb{R}^n)$, are regular, localised (which we've just shown) and the k^{th} moments vanish for $0 \leq k \leq r$).

Chapter 5

Unconditional Bases and Khinchin's Inequality

Definition 5.1 *Let $\{x_j\}_{j\in\mathbb{N}}$ be a sequence in a normed linear space.*

1. *The sequence is said to be unconditionally summable if for any permutation σ of $\mathbb{N}$, $\sum x_{\sigma(j)}$ converges.*

2. *The sequence is said to be sign summable if for any sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of 1s and -1s, $\sum \varepsilon_j x_j$ converges.*

3. *The sequence is said to be fishy summable if there exists a constant C such that for any finite subset $F \subset \mathbb{N}$, and for any sequence $\varepsilon(\lambda), \lambda \in F$ of 1s and -1s, $\left\|\sum_{\lambda\in F} \varepsilon(\lambda)x_\lambda\right\| \leq C$.*

Theorem 5.2 *Suppose $\{x_k\}$ is a sequence in a Banach space, then the following are equivalent*

1. *$\{x_k\}$ is unconditionally summable;*

2. *$\{x_k\}$ is sign summable;*

3. *$\{x_k\}$ is fishy summable.*

The reader is referred to [4] for the proof.

Definition 5.3 *Let $\{e_k\}$ be a sequence of vectors in a Banach space X. The sequence is said to be an unconditional basis of X if*

- *For any vector $x \in X$, there exists a unique sequence of scalars $\{\alpha_k\}$ such that $x = \sum \alpha_k e_k$ (with convergence of the sum occurring in the Banach space norm);*
- *The series above is unconditionally convergent.*

5.1 Khinchin's Inequality

Definition 5.4 *For $n \in \mathbb{N}$, we define the Rademacher function, $r_n : [0,1] \longmapsto \mathbb{R}$ by*

$$r_n(t) = sign\left(\sin\left(2^n \pi t\right)\right)$$

and by definition $r_0 \equiv 1$. Alternatively, one has

$$r_n(t) = \sum_{2^n \leq l < 2^{n+1}} q_l(t)$$

where $q_l\left(t\right) = 2^{-n/2} h_l(t)$ and h_l is the l^{th} Haar function.

Lemma 5.5 *The collection of Rademacher functions are a set of independent random variables on $([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ are the Borel sets of $[0,1]$, and λ is the restriction of the Lebesgue measure to $\mathcal{B}$.*

Proof. For each $j \in \mathbb{N}$, define $l_j = 2^{-j}$, and let A_{jk} be the interval $((2k-1)l_j, 2kl_j]$. Define $B_j = \coprod_{1 \leq k \leq 2^{j-1}} A_{jk}$.

For any real number x, and any non-negative integer n, the event

$$(r_n \leq x) = \begin{cases} [0,1] & x \geq 1 \\ B_n & -1 < x \leq 1 \\ \emptyset & x < -1 \end{cases}$$

so we merely need to show that the events B_j are independent. Observe that for all j, $\lambda\left(B_j\right) = \frac{1}{2}$, so it suffices to prove that if $0 < n_1 < \ldots < n_m$ are integers, then $\lambda(\bigcap_{1 \leq j \leq m} B_{n_j}) = 2^{-k}$. I'll prove the case $\lambda\left(B_n \cap B_m\right) = \frac{1}{4}$, the general case is similar. Suppose $n < m$. One can easily verify that $B_n \cap B_m = \coprod A_{m\,k_j}$, where the disjoint union is taken over all k_j such that $A_{m\,k_j} \subseteq A_{n\,k}$ for some k. One can also see (draw a picture!) that the number of such k_j is $2^{n-1}2^{m-n-1} = 2^{m-2}$. And $\lambda\left(\coprod A_{m\,k_j}\right) = \#k_j.\lambda\left(A_{m\,k_j}\right) = 2^{m-2}(2^{-1}2^{1-m}) = 2^{-2} = \frac{1}{4}$, as required. ■

Proposition 5.6 *Suppose $0 \leq p_1, \ldots, p_k$ and $0 < n_1 < \ldots < n_k$ are integers, then*

$$\int_0^1 \prod_{j=1}^k r_{n_j}^{p_j}(t)dt = \begin{cases} 1 & \text{if each } p_j \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

in particular, the Rademacher functions are orthonormal.

Proof. Observe that for any integer n, if p is even, then $r_n^p(t) \equiv 1$. Otherwise, if p is odd, $r_n^p(t) = r_n(t)$. This immediately gives

$$\int_0^1 \prod_{j=1}^k r_{n_j}^{p_j}(t)dt = 1, \text{ if each } p_j \text{ is even.}$$

and

$$\int_0^1 \prod_{j=1}^k r_{n_j}^{p_j}(t)dt = \int_0^1 \prod_{j_i=1}^{k_i} r_{n_{j_i}}(t)dt$$

where n_{j_i} is the subsequence of n_j such that each p_{j_i} is odd. So, without loss of generality, we may consider

$$\int_0^1 \prod_{j=1}^k r_{n_j}(t)dt$$

but by the independence of the Rademacher functions, we have

$$\int_0^1 \prod_{j=1}^k r_{n_j}(t)dt = \prod_{j=1}^k \int_0^1 r_{n_j}(t)dt = 0$$

■

Theorem 5.7 (Khinchin's Inequality) *If $0 < p < \infty$, then there exist constants $0 < A_p \leq B_p$, such that for any sequence $\{a_n\}_{n\in\mathbb{N}} \in l_2$,*

$$A_p\left(\sum_n |a_n|^2\right)^{1/2} \leq \left(\int_0^1 |\sum_n a_n r_n(t)|^p\right)^{1/p} \leq B_p\left(\sum_n |a_n|^2\right)^{1/2}$$

Proof. Firstly, suppose $p \in \mathbb{N}$. Set $f(t) = \sum_{n=1}^{m} a_n r_n(t)$, (we only deal with finite sums, the result holds for infinite sums by taking limits). Then, using the result that for any non-negative real α, $\alpha^p < p!(1 + \frac{\alpha^p}{p!}) \leq p! e^{\alpha}$, we have

$$\int_0^1 |f(t)|^p \, dt \leq \int_0^1 p! e^{|f(t)|} dt \leq p! \int_0^1 \left(e^{f(t)} + e^{-f(t)}\right) dt$$

now, using the independence of the Rademacher functions, we have:

$$\int_0^1 e^{f(t)} dt = \int_0^1 \prod_{n=1}^{m} \exp\left(a_n r_n(t)\right) dt = \prod_{n=1}^{m} \int_0^1 \exp\left(a_n r_n(t)\right) dt = \prod_{n=1}^{m} \cosh(a_n)$$

where for the last equality, we have used the fact that $\exp\left(r_n(t)\right)$ is piecewise constant, and hence

$$\int_0^1 \exp\left(r_n(t)\right) dt = \exp(1)\lambda(r_n = 1) + \exp(-1)\lambda(r_n = -1) = \frac{e^1 + e^{-1}}{2} = \cosh 1$$

now observe that for $n \in \mathbb{N}$, $2n.(2n-1).(2n-2)\ldots(n+1) \geq 2^n$, so $(2n)! \geq n! 2^n$, and hence

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \leq \sum_{n=0}^{\infty} \frac{x^{2n}}{n! 2^n} = \exp(\frac{x^2}{2})$$

using this in the above relation, we have

$$\int_0^1 e^{f(t)} dt \leq \prod_{n=1}^{m} \exp(\frac{a_n^2}{2}) = \exp\left(\sum_{n=1}^{m} \frac{a_n^2}{2}\right)$$

but by the orthonormality of the Rademacher functions,

$$\|f\|_2 = \left\|\sum a_n r_n\right\|_2 = \left(\sum_{n=1}^{m} a_n^2\right)^{1/2}$$

Assume that f is normalised so that $\|f\|_2 = 1$, then

$$\int_0^1 e^{f(t)} dt \leq e^{1/2}$$

and symmetrically, $\int_0^1 e^{-f(t)}dt \le e^{1/2}$, so that we arrive at

$$\|f\|_p \le \left(2p!e^{1/2}\right)^{1/p}$$

and for general f (not necessarily of unit norm), put $g = \frac{f}{\|f\|_2}$, so $\|g\|_2 = 1$, and

$$\begin{aligned} \frac{\|f\|_p}{\|f\|_2} = \|g\|_p &\le (2p!e^{1/2})^{1/p} \\ \|f\|_p &\le B_p \|f\|_2 \end{aligned}$$

as required, where $B_p = (2p!e^{1/2})^{1/p}$.

So we've proved the right-hand side of Khinchin's Inequality for integer p. Now let p be an arbitrary real such that $2 \le p < \infty$. We get the left-hand side of the inequality because the L_p norms are monotone on a probability space, and choosing $k = \ulcorner p \urcorner$,

$$\|f\|_p \le \|f\|_k \le (2k!e^{1/2})^{1/k} \|f\|_2$$

and we have both sides of the inequality for $p \ge 2$. What I really need for the wavelets is to prove the inequality with $p = 1$, which we finally come to!

Monotonicity takes care of the right-hand side and we just have to show the left-hand side. Observe that $2 = \frac{2}{3} + 4(\frac{1}{3})$, and hence

$$\int_0^1 |f(t)|^2 \, dt = \int_0^1 |f(t)|^{\frac{2}{3}} . |f(t)|^{4(\frac{1}{3})} \, dt \le \left(\int_0^1 |f(t)| \, dt\right)^{\frac{2}{3}} \left(\int_0^1 |f(t)|^4 \, dt\right)^{\frac{1}{3}}$$

where we have employed the Hölder inequality with '$p = \frac{3}{2}$' and '$q = 3$'. But $\|f\|_4 \le B_4 \|f\|_2$, and we get

$$\begin{aligned} \|f\|_2 &\le \|f\|_1^{2/3} (B_4 \|f\|_2)^{1/3} \\ \|f\|_2 &\le \|f\|_1 B_4^2 \end{aligned}$$

Note that this trick can be used to prove Khinchin's Inequality for all p with $0 < p < 2$. ■

Corollary 5.8 *For any sequence* $a_1, \ldots, a_n$ *of complex numbers,*

$$(\sum_1^n |a_j|^2)^{1/2} \leq C 2^{-n} \sum_{\varepsilon_1} \cdots \sum_{\varepsilon_n} |\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n|$$

Proof. Let $f(t) = \sum_{j=1}^n a_j r_j(t)$, then using the fact that $f(t)$ is piecewise constant (and other properties, draw a picture), we have

$$\begin{aligned}
\|f\|_1 &= \int_0^1 \left|\sum a_j r_j(t)\right| dt \\
&= 2^{-n} \sum_{k=0}^{2^n-1} |f(t)|_{[k2^{-n},(k+1)2^{-n})} \\
&= 2^{-n} \sum_{\varepsilon_1} \cdots \sum_{\varepsilon_n} |\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n|
\end{aligned}$$

where it's easiest to see the last relation by drawing another picture. Applying Khinchin's Inequality, we have

$$A_1 \|f\|_2 \leq \|f\|_1 = 2^{-n} \sum_{\varepsilon_1} \cdots \sum_{\varepsilon_n} |\varepsilon_1 a_1 + \ldots + \varepsilon_n a_n|$$

whence the result follows, with $C = A_p^{-1}$. ■

Chapter 6

Wavelets and the Hardy space $H^1(\mathbb{R}^n)$

We now come to the main topic of this article, the analysis of the Hardy space $H^1(\mathbb{R}^n)$ by the use of wavelet methods. From the point of view of wavelets, $H^1(\mathbb{R}^n)$ is a subspace of $L^1(\mathbb{R}^n)$ for which the regular wavelets of compact support form an unconditional basis, but there is also a classical 'atomic' definition of $H^1(\mathbb{R}^n)$ which we will show to be equivalent to the wavelet definition.

Definition 6.1 *Atomic $H^1(\mathbb{R}^n)$ is the subspace of $L^1(\mathbb{R}^n)$ consisting of functions f of the form $f(x) = \sum_0^\infty \lambda_j a_j(x)$, where $\lambda = \{\lambda_j\} \in l^\infty$ and $\{a_j\}$ is a sequence of atoms, where an atom $a(x)$ is by definition an element of $L^2(\mathbb{R}^n)$ such that the following conditions are satisfied:*

- *The support of a is contained in B for some ball $B \subset \mathbb{R}^2$;*
- $\|a\|_2 \leq |B|^{-1/2}$;
- $\int_B a(x) = 0$.

(Note that for such a, $\|a\|_1 = \int_B a.1dx \leq \|a\|_2 |B|^{1/2} \leq 1$, so we have $a \in L^1(\mathbb{R}^n)$).

Given a function f in atomic $H^1(\mathbb{R}^n)$, we define its norm to be the infimum of $\|\lambda\|_1$ over all possible atomic decompositions of f.

Our first theorem will relate the atomic $H^1(\mathbb{R}^n)$ space to a condition related to the unconditional wavelet basis definition of $H^1(\mathbb{R}^n)$. First some definitions:

Definition 6.2 *Let $j \in \mathbb{Z}$ and let $k \in \mathbb{Z}^n$, we define the dyadic cube $Q(j,k) = \{x \in \mathbb{R}^n : 2^j x - k \in [0,1)^n\}$. The set of all dyadic cubes is denoted by $\mathcal{Q}$.*

Definition 6.3 *Let α be a function from $\mathcal{Q}$ to $\mathbb{C}$. We say α belongs to the tent space T_1 if the function $S(\alpha) \in L^1(\mathbb{R}^n)$, where by definition,*

$$S(\alpha)(x) = \left(\sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_Q(x) \right)^{1/2}$$

and if $\alpha \in T_1$, we define its norm by $\|\alpha\|_{T_1} = \|S(\alpha)\|_1$.

Definition 6.4 *A function $\alpha : \mathcal{Q} \to \mathbb{C}$ is said to be an atom if there exists a dyadic cube R such that $\alpha(Q) = 0$ whenever $Q \cap R = \emptyset$, and*

$$\sum |\alpha(Q)|^2 \leq |R|^{-1}$$

the smallest such R is called the base of the atom.

Remark 6.5 *It is important to realise that we have two different kinds of atoms here, atoms in T_1 (which are sequences on $\mathcal{Q}$), and atoms in $H^1(\mathbb{R}^n)$, which are functions on $\mathbb{R}^n$. However, we'll show that there is indeed a link between the two concepts.*

Theorem 6.6 *Let α be a function from $\mathcal{Q}$ to $\mathbb{C}$. Then the following are equivalent:*

1. *$\alpha \in T_1$;*

2. *α admits a decomposition by atoms α_j from the tent space T_1, that is, there exists a sequence $\{\alpha_j\}_{j=0}^{\infty}$ of atoms of T_1, and a sequence $\lambda = \{\lambda_j\}_{j=0}^{\infty}$ of scalars such that $\lambda \in l^1$ and $\alpha = \sum_{j=0}^{\infty} \lambda_j \alpha_j$;*

3. *there exists $\gamma \in [0,1)$, such that for any $Q \in \mathcal{Q}$, there exists a measurable set $R \subset Q$ such that $|Q| \leq \gamma |R|$ and*

$$\left(\sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \right)^{1/2} \in L^1(\mathbb{R}^n)$$

The implication (1)$\Longrightarrow$(3) is trivial, we simply set $\gamma = 1$ and $R = Q$. Now we show (2)$\Longrightarrow$(1):

Suppose α is an atom with base R, then we have that

$$\begin{aligned} \|a\|_{T_1} &= \int S(\alpha)\,dx = \int_R S(\alpha)\,.1dx \leq \left(\int (S(\alpha))^2\,dx \right)^{1/2} |R|^{1/2} \\ &= |R|^{1/2} \left(\int \sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_Q(x)\,dx \right)^{1/2} \\ &= |R|^{1/2} \left(\sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 \right)^{1/2} \\ &\leq 1 \end{aligned}$$

where in the last line we've used the definition of an atom.

Now if α is any function on $\mathcal{Q}$ with an atomic decompostion as in the statement of the theorem, then

$$\|\alpha\|_{T_1} = \left\| \sum_0^\infty \lambda_j \alpha_j \right\|_{T_1} \leq \sum_0^\infty |\lambda_j| \|\alpha_j\|_{T_1} \leq \sum_0^\infty |\lambda_j| < \infty$$

hence $\alpha \in T_1$.

The part that requires some work is the proof that (3)$\Longrightarrow$(2).

Suppose then that (3). Define

$$\sigma(x) = \left(\sum_{Q \in \mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \right)^{1/2}$$

so we have $\sigma \in L^1(\mathbb{R}^n)$. To start off on our atomic decomposition, we will define sets E_k by$E_k = \left\{ x : \sigma(x) > 2^k \right\}$, where $k \in \mathbb{Z}$.These will be used to determine the support of α (where by definition, the support of α is the set of $Q \in \mathcal{Q}$ for which $\alpha(Q) \neq 0$), and to get the bases for the atoms in the atomic decomposition. But the E_k are themselves fairly arbitrary sets, so we need some way of cubifying them. Fix some constant $\beta \in (0, \gamma)$, then pick $k \in \mathbb{Z}$. We define the collection $\mathcal{C}_k$ to be the set of dyadic cubes which are nearly contained in E_k, that is,

$$\mathcal{C}_k = \{Q \in \mathcal{Q} : |Q \cap E_k| \geq \beta\, |Q|\}$$

and we then define the magnification E_k^* of E_k by $E_k^* = \bigcup_{Q \in \mathcal{C}_k} Q$.

Now, to get maximality, observe that the elements of $\mathcal{C}_k$ have their volume bounded from above (suppose $Q \in \mathcal{C}_k$ then $|Q| \leq \beta^{-1} |Q \cap E_k| \leq \beta^{-1} |E_k|$) and that if any distinct pair of dyadic cubes have non-empty intersection, then one must be contained in the other and the volume of the larger one is at least twice that of the smaller one. From the Hausdorff maximal principle, each $Q \in \mathcal{C}_k$ is contained in some maximal chain, and the above comments show that this chain has a maximal element, say $Q(k, l) \in \mathcal{C}_k$.

The $Q\,(k, l)$ are disjoint, as l varies (say $Q\,(k, l_1)$ and $Q\,(k, l_2)$ have non-empty intersection; without loss of generality, assume $Q\,(k, l_1) \subset Q\,(k, l_2)$, then by maximality of $Q\,(k, l_1)$, $Q\,(k, l_1) = Q\,(k, l_2)$). It is also clear that each $Q \in \mathcal{C}_k$ is contained in some $Q\,(k, l) \in \mathcal{C}_k$. That is, the sets $Q\,(k, l)$, as l varies, form a partition of $\mathcal{C}_k$.

Denote the support of α by $\mathcal{C}$. We'll show that $\mathcal{C} \subset \bigcup_k \mathcal{C}_k$. For let $Q' \in \mathcal{Q}$ such that $\alpha\,(Q') \neq 0$, there exists $k \in \mathbb{Z}$ small enough such that $|\alpha\,(Q')|\, |Q'|^{-1/2} > 2^k$. Suppose $R' \subset Q'$ with $|R'| \geq \gamma\, |Q'|$ and that $x \in R'$, then

$$\begin{aligned} \sigma(x) &= \left(\sum_{Q \in \mathcal{Q}} |\alpha\,(Q)|^2\, |Q|^{-1}\, \chi_R\,(x) \right)^{1/2} \\ &\geq \left(\left|\alpha\,(Q')\right|^2 \left|Q'\right|^{-1} \right)^{1/2} \\ &> 2^k \end{aligned}$$

so $E_k \supset R'$, from which we get $Q' \cap E_k \supset R'$, then $|Q' \cap E_k| \geq |R'| \geq \gamma\, |Q'| > \beta\, |Q|$, consequently $Q' \in \mathcal{C}_k$.

Define $\Delta_k = \mathcal{C}_k \backslash \mathcal{C}_{k+1}$, (so that Δ_k is approximately is the set of Q for which Q is nearly $\{x : 2^k < \sigma(x) \leq 2^{k+1}\}$). Define $\Delta(k,l) = \{Q \in \Delta_k : Q \subset Q(k,l)\}$. The $\Delta(k,l)$ will be the bases of the atoms in our atomic decomposition. Fix $k \in \mathbb{Z}$, then the set of $\Delta(k,l)$ as l varies forms a partition of Δ_k. (Reason: certainly if $Q \in \Delta(k,l)$, then by definition $Q \in \Delta_k$, and if $Q \in \Delta(k,l) \cap \Delta(k,l')$, then $Q \in Q(k,l)$ and $Q \in Q(k,l')$, but, as we've shown, this implies $Q(k,l) = Q(k,l')$).

Suppose, more generally, that $Q \in \Delta(k,l) \cap \Delta(k',l')$. Now if $k < k'$, then $E_k \supset E_{k'}$ (by definition of E_k), hence $\mathcal{C}_k \supset \mathcal{C}_{k'}$ (for suppose $Q' \in \mathcal{C}_{k'}$, then $|Q' \cap E_k| \geq |Q' \cap E_{k'}| \geq \beta |Q'|$, hence $Q' \in \mathcal{C}_k$). But this implies that the Δ_k are disjoint, and hence $Q = \emptyset$. Consequently $\mathcal{C} \subset \coprod_{k,l} \Delta(k,l)$.

Now we're going to construct atoms $\alpha_{k,l}$ with bases contained in $Q(k,l)$ and constants $\lambda(k,l)$ such that $\alpha = \sum_{k,l} \lambda(k,l) \alpha_{k,l}$.

Indeed, define $\lambda(k,l) = |Q(k,l)|^{1/2} \left(\sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 \right)^{1/2}$ and

$$\alpha_{k,l}(Q) = \begin{cases} (\lambda(k,l))^{-1} \alpha(Q) & Q \in \Delta(k,l) \\ 0 & \text{otherwise} \end{cases}$$

by construction, the base of $\alpha_{k,l}$ is contained in $Q(k,l)$, moreover,

$$\begin{aligned} \sum_{k,l} \lambda(k,l) \alpha_{k,l}(Q) &= \sum_{\Delta(k,l) \ni Q} \lambda(k,l) (\lambda(k,l))^{-1} \alpha(Q) \\ &= \sum_{\Delta(k,l) \ni Q} \alpha(Q) = \alpha(Q) \end{aligned}$$

where in the last equation, I've used the fact that the $\Delta(k,l)$s form a partition of the support of α. We immediately have that the $\alpha_{k,l}$ are atoms because

$$\begin{aligned} \sum_{Q \in \mathcal{Q}} |\alpha_{k,l}(Q)|^2 &= \sum_{Q \in \Delta(k,l)} \left| \alpha(Q) \lambda(k,l)^{-1} \right|^2 \\ &= \sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 |Q(k,l)|^{-1} \left(\sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 \right)^{-1} \\ &= |Q(k,l)|^{-1} \end{aligned}$$

What is left to show is that $\sum_{k,l} \lambda(k,l) < \infty$. This will take some time, and will hinge on the double inequality:

$$\sum_{Q \in \Delta(k,l)} |\alpha(Q)|^2 \le \frac{1}{\gamma - \beta} \int_{Q(k,l) \setminus E_{k+1}} \sigma^2(x)\, dx \le \frac{1}{\gamma - \beta} 4^{k+1} |Q(k,l)| \tag{6.1}$$

We'll prove the inequality later, but assuming it for now, we have

$$\begin{aligned} \sum_{k,l} \lambda(k,l) &\le \sum_{k,l} |Q(k,l)|^{1/2} \left[\frac{1}{\gamma - \beta} 4^{k+1} |Q(k,l)|\right]^{1/2} \\ &= 2(\gamma - \beta)^{-1/2} \sum_k \sum_l 2^k |Q(k,l)| \end{aligned}$$

Now use the fact that because each $Q(k,l) \in \mathcal{C}_k$ and are disjoint,

$$\sum_l |Q(k,l)| \le \beta^{-1} \sum_l |Q(k,l) \cap E_k| = \beta^{-1} \left| \coprod_l Q(k,l) \cap E_k \right| \le \beta^{-1} |E_k|$$

Using this in the above inequality,

$$\sum_{k,l} \lambda(k,l) \le 2[\beta(\gamma - \beta)]^{-1/2} \sum_k 2^k |E_k|$$

then apply lemma 6.7 to get

$$\le 4[\beta(\gamma - \beta)]^{-1/2} \|\sigma\|_1$$

So we've demonstrated the existence of a (finite) constant $C = 4[\beta(\gamma - \beta)]^{-1/2}$ such that

$$\sum_{k,l} \lambda(k,l) \le C \|\sigma\|_1$$

concluding the proof of existence of the atomic decomposition of α.

Well almost. I still have to demonstrate the double inequality 6.1. For the second inequality, we know that if $x \notin E_{k+1}$, then $\sigma(x) \le 2^{k+1}$, so

$$\begin{aligned} \int_{Q(k,l) \setminus E_{k+1}} \sigma^2(x)\, dx &\le \left(2^{k+1}\right)^2 |Q(k,l) \setminus E_{k+1}| \\ &\le 4^{k+1} |Q(k,l)| \end{aligned}$$

For the first inequality, note

$$\begin{aligned}\sigma^2(x) &= \sum_{Q\in\mathcal{Q}} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x) \\ &\geq \sum_{Q\in\Delta(k,l)} |\alpha(Q)|^2 |Q|^{-1} \chi_R(x)\end{aligned}$$

so

$$\begin{aligned}\int_{Q(k,l)\backslash E_{k+1}} \sigma^2(x)\,dx &\geq \sum_{Q\in\Delta(k,l)} |\alpha(Q)|^2 |Q|^{-1} |R\cap(Q(k,l)\backslash E_{k+1})| \\ &= \sum_{Q\in\Delta(k,l)} |\alpha(Q)|^2 |Q|^{-1} |R\backslash E_{k+1}|\end{aligned}$$

where in the last equation, I've used the fact that $Q \in \Delta(k,l)$ implies $Q \subset Q(k,l)$ and hence $R \subset Q(k,l)$. To finish, we need to estimate $|R\backslash E_{k+1}|$. But we have $|R\backslash E_{k+1}| = |R| - |R\cap E_{k+1}| \geq |R| - |Q\cap E_{k+1}|$. Because $Q\in\Delta(k,l)$, we have $Q \notin \mathcal{C}_{k+1}$, hence $|Q\cap E_{k+1}| < \beta|Q| \leq \frac{\beta}{\gamma}|R|$, whence $|R\backslash E_{k+1}| > |R|(1-\frac{\beta}{\gamma}) = |R|\gamma^{-1}(\gamma-\beta) > |Q|(\gamma-\beta)$. Using this in the above displayed equation gives the desired result.

Lemma 6.7 *Let $\sigma(x)$ be a non-negative measurable function on $\mathbb{R}^n$. For $k\in\mathbb{Z}$, define $E_k = \{x : \sigma(x) > 2^k\}$, then*

$$\sum_{-\infty}^{\infty} 2^k |E_k| \leq 2\int_{\mathbb{R}^n} \sigma(x)\,dx$$

Proof. I'll prove the result for any bounded simple function σ, then the general result holds by approximating a measurable function by a simple function.

Write $\sigma = \sum_{i=1}^N a_i\chi_i$, where the χ_i are characteristic functions of measurable sets A_i write $l_j = |A_j|$, we may assume that the a_i are increasing. Put $b_i = \lceil \log_2(a_i)\rceil$. Then

$$\begin{aligned}\sum_{-\infty}^{\infty} 2^k|E_k| &= \sum_{k=-\infty}^{b_1-1} 2^k\left(\sum_{j=1}^N l_j\right) + 2^{b_1}\left(\sum_{a_j>2^{b_1}} l_j\right) + 2^{b_1+1}\left(\sum_{a_j>2^{b_1+1}} l_j\right) + \ldots \\ &= l_1\left(\sum_{k=-\infty}^{b_1-1} 2^k\right) + l_2\left(\sum_{k<b_2} 2^k\right) + \ldots + l_N\left(\sum_{k<b_N} 2^k\right) \\ &= 2^{b_1}l_1 + 2^{b_2}l_2 + \ldots + 2^{b_N}l_N\end{aligned}$$

$$\leq 2(a_1 l_1 + a_2 l_2 + \ldots + a_N l_N)$$
$$= 2 \int_{\mathbb{R}^n} \sigma(x)\, dx$$

■

6.1 A fistful of norms

And now for the grand finale. I will show that the atomic definition of $H^1(\mathbb{R}^n)$ is equivalent to the wavelet definition. That is

Theorem 6.8 *Let $f(x) = \sum_{\lambda \in \Lambda} \alpha(\lambda)\, \psi_\lambda(x) \in L^1(\mathbb{R}^n)$, then its wavelet series converges unconditionally if and only if f has an atomic decomposition.*

The way we interpret the series is that Λ is the set of all dyadic positions $\lambda = k2^{-j} + \varepsilon 2^{-j-1}$, for $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, and $\varepsilon \in E$ as defined in section 4.2 is a sequence of signs which serves as an index into our set of basic wavelets. Put $Q(\lambda) = Q(j,k)$ (as defined in 6.2), then $\psi_\lambda := \psi^\varepsilon_{Q(\lambda)} := 2^{nj/2}\psi^\varepsilon\left(2^j \cdot -k\right)$.

Now by theorem 5.2, saying that the wavelet series converges unconditionally is equivalent to saying that

$$\sup_{\substack{F \text{ finite} \\ F \subset \Lambda}} \sup_{\varepsilon(\lambda) = \pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda)\, \alpha(\lambda)\, \psi_\lambda(x) \right\|_1 < \infty$$

so we will prove theorem 6.8 by the following set of equivalences:

Theorem 6.9 *Let $f(x) = \sum_{\lambda \in \Lambda} \alpha(\lambda)\, \psi_\lambda(x)$. Then the following are equivalent:*

1. $\sup_{\substack{F \text{ finite} \\ F \subset \Lambda}} \sup_{\varepsilon(\lambda)=\pm 1} \left\| \sum_{\lambda \in F} \varepsilon(\lambda)\, \alpha(\lambda)\, \psi_\lambda(x) \right\|_1 < \infty$;
2. $\left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2\right)^{1/2} \in L^1(\mathbb{R}^n)$;
3. *there exists $\gamma \in [0,1)$, such that for any $\lambda \in \Lambda$, there exists a measurable set $R(\lambda) \subset Q(\lambda)$ such that $\gamma |Q(\lambda)| \leq |R(\lambda)|$ and*

$$\left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |Q(\lambda)|^{-1} \chi_{R(\lambda)}(x) \right)^{1/2} \in L^1(\mathbb{R}^n)$$

4. $\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2|Q(\lambda)|^{-1}\chi_{Q(\lambda)}(x)\right)^{1/2}\in L^1(\mathbb{R}^n)$

5. f has an atomic decomposition.

(4)⇒(3): As before, this is trivial. Put $\gamma=1$, $R(\lambda)=Q(\lambda)$.

(1)⇒(2): Define $a_\lambda(x)=\alpha(\lambda)\psi_\lambda(x)$, we have

$$\begin{aligned}\int\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2|\psi_\lambda(x)|^2\right)^{1/2}dx &\leq \int\left(\sum|a_\lambda(x)|^2\right)^{1/2}dx\\ &\leq C2^{-n}\int\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_n}|\varepsilon_1a_1+\ldots+\varepsilon_na_n|\,dx\end{aligned}$$

(where we have used the corollary to Khinchin's inequality.)

$$\leq C2^{-n}\sum_{\varepsilon_1}\cdots\sum_{\varepsilon_n}C'<\infty$$

where $C'=\sup\limits_{\substack{F\text{ finite}\\F\subset\Lambda}}\sup\limits_{\varepsilon(\lambda)=\pm1}\left\|\sum_{\lambda\in F}\varepsilon(\lambda)\,\alpha(\lambda)\,\psi_\lambda(x)\right\|_1$.

(2)⇒(3): One can easily show that if ψ^ε is a basic wavelet, then any transalted version of ψ^ε is also a basic wavelet and by continuity of the ψ^ε, we have that there exist constants $\gamma,c>0$ such that for any $\varepsilon\in E$, there exists a cube $A^\varepsilon\subset[0,1)^n$ such that $|A^\varepsilon|\geq\gamma$ and $|\psi^\varepsilon(x)|\geq c$ whenever $x\in A^\varepsilon$ (where, if necessary, we have replaced each ψ^ε by an appropriate translated version). Now we define the cubes $R(\lambda)$ by $R(\lambda)=\left\{x:2^jx-k\in A^\varepsilon\right\}$ (where, as always, $\lambda=k2^{-j}+\varepsilon2^{-j-1}$).So certainly $|R(\lambda)|\geq\gamma|Q(\lambda)|$, and for each $x\in R(\lambda)$, $|\psi_\lambda(x)|=\left|2^{nj/2}\psi^\varepsilon\left(2^jx-k\right)\right|\geq c2^{nj/2}=c\left(2^{-nj}\right)^{-1/2}=c|Q|^{-1/2}$. That is, on $R(\lambda)$, $|Q|^{-1}\leq c^{-2}|\psi_\lambda(x)|^2$.So we have

$$\begin{aligned}\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2|Q(\lambda)|^{-1}\chi_{R(\lambda)}(x)\right)^{1/2} &\leq \left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2c^{-2}|\psi_\lambda(x)|^2\chi_{R(\lambda)}(x)\right)^{1/2}\\ &\leq c^{-1}\left(\sum_{\lambda\in\Lambda}|\alpha(\lambda)|^2|\psi_\lambda(x)|^2\right)^{1/2}\end{aligned}$$

by hypotheses, the right-hand side is in $L^1(\mathbb{R}^n)$, hence also the left-hand side is.

(3)⇒(5): Think of the function $\alpha : \Lambda \to \mathbb{C}$ as a sequence. Then it is determined by the $2^n - 1$ sequences $\alpha^\varepsilon : \mathcal{Q} \to \mathbb{C}$. Then

$$\left(\sum_{Q \in \mathcal{Q}} \left| \alpha^\varepsilon (Q) \right|^2 \left| Q \right|^{-1} \chi_R (x) \right)^{1/2} \leq \left(\sum_{\lambda \in \Lambda} \left| \alpha (\lambda) \right|^2 \left| Q (\lambda) \right|^{-1} \chi_{R(\lambda)} (x) \right)^{1/2}$$

the right-hand side is in $L^1 (\mathbb{R}^n)$, so the left-hand side is. We apply theorem 6.6 and see that the *sequence* α^ε has a decomposition by atoms of T_1, say $\alpha^\varepsilon (Q) = \sum_{i=0}^\infty \lambda_i^\varepsilon \alpha_i^\varepsilon (Q)$, where $\sum_{i=0}^\infty \lambda_i < \infty$ and the α_i^ε are atoms of T_1. Of course, we want to show that $f(x) = \sum_{\lambda \in \Lambda} \alpha (\lambda) \psi_\lambda (x)$ has a decomposition by atoms of $H^1 (\mathbb{R}^n)$. Indeed, we will show that the functions $a_i^\varepsilon (x) = \sum_{Q \in \mathcal{Q}} \alpha_i^\varepsilon (Q) \psi_Q^\varepsilon (x)$ are atoms of $H^1 (\mathbb{R}^n)$ (well, they're nearly atoms, we need to tweak them by a uniform constant), then we'd have

$$\begin{aligned} f(x) &= \sum_{\lambda \in \Lambda} \alpha (\lambda) \psi_\lambda (x) \\ &= \sum_{\varepsilon \in E} \sum_{Q \in \mathcal{Q}} \alpha^\varepsilon (Q) \psi_Q^\varepsilon (x) \\ &= \sum_{\varepsilon \in E} \sum_{Q \in \mathcal{Q}} \sum_{i=0}^\infty \lambda_i^\varepsilon \alpha_i^\varepsilon (Q) \psi_Q^\varepsilon (x) \\ &= \sum_{\varepsilon \in E} \sum_{i=0}^\infty \lambda_i^\varepsilon a_i^\varepsilon (x) \end{aligned}$$

which is then an atomic decomposition of f (well, a finite sum of atomic decompositions, which amounts to the same thing).

For the first time, we use the fact that our basic wavelets ψ^ε are compactly supported. As in section 4.2, we know that there exists a constant $m \geq 1$ such that for each $Q \in \mathcal{Q}$, and each $\varepsilon \in E$, the support of ψ_Q^ε is contained in mQ. Define Q_i^ε to be the base of the atom $\alpha_i^\varepsilon (Q)$. Then $a_i^\varepsilon (x) = \sum_{Q \in \mathcal{Q}} \alpha_i^\varepsilon (Q) \psi_Q^\varepsilon (x)$ has support which is contained in mQ_i^ε also$\| a_i^\varepsilon \|_2 = \left(\sum_{Q \in \mathcal{Q}} \left| \alpha_i^\varepsilon (Q) \right|^2 \right)^{1/2}$ (by orthonormality of the wavelets!) $\leq \left| Q_i^\varepsilon \right|^{-1/2}$ (by definition of an atom in T_1). Equally, $\left| mQ_i^\varepsilon \right|^{-1/2} = m^{-n/2} \left| Q_i^\varepsilon \right|^{-1/2} \geq \left\| m^{-n/2} a_i^\varepsilon \right\|_2$.

Remark 6.10 *This doesn't quite prove atomicity in $H^1 (\mathbb{R}^n)$, as we've used dyadic cubes instead of balls; however each cube Q is contained in a unique minimal ball B, and the ratio*

$\zeta_n = |B| / |Q|$ *is independent of the choice of cube. If a function* $\mathfrak{a} \in L^2(\mathbb{R}^n)$ *satisfies the conditions that for some cube* Q*, the support of* $\mathfrak{a}$ *is contained in* Q *and* $\|\mathfrak{a}\|_2 \leq |Q|^{-1/2}$ *and* $\int \mathfrak{a} dx = 0$*. Then we also have that the support of* $\mathfrak{a}$ *is contained in* B*, and* $|B|^{-1/2} = \zeta_n^{-1/2} |Q|^{-1/2} \geq \left\| \xi_n^{-1/2} \mathfrak{a} \right\|_2$

Because a_i^ε has compact support and is in $L^2(\mathbb{R}^n)$, it is in $L^1(\mathbb{R}^n)$, and the Lebesgue dominated convergence theorem applies,

$$\int a_i^\varepsilon(x)\, dx = \sum_{Q \in \mathcal{Q}} \alpha_i^\varepsilon(Q) \int \psi_Q^\varepsilon(x)\, dx = 0$$

Hence by the remarks, we know that $m^{-n/2} \zeta^{-1/2} a_i^\varepsilon$ is an atom of $H^1(\mathbb{R}^n)$.

(3)⇒(4): As in the proof of the previous part of the theorem, we know that each of the sequences α^ε belongs to the tent space T_1. Consequently, the same is true for the sequence α, and the result follows.

(5)⇒(2): We want to show that if f has an atomic decomposition, then

$$Tf(x) := \left(\sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right)^{1/2}$$

is Lebesgue integrable. First, let's show the result holds if f is an atom of $H^1(\mathbb{R}^n)$, then the result will follow for any f with an atomic decomposition.

An initial observation is that by Pythagoras, $\|f\|_2^2 = \sum_{\lambda \in \Lambda} |\alpha(\lambda)|^2$, and

$$\|Tf\|_2^2 = \int \left(\sum_\lambda |\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 \right) dx = \sum_\lambda \left[|\alpha(\lambda)|^2 \int |\psi_\lambda(x)|^2\, dx \right] = \|f\|_2^2$$

Suppose f has support contained in the ball $B = \{x : |x - x_0| \leq r\}$. Choose $C > 1$. We make $\mathbb{R}^n$ into an onion centered about x_0 by defining the core of the onion to be the ball $\tilde{B} = CB$, the rest of $\mathbb{R}^n$ is then decomposed into layers $R_k = \left\{x : 2^k Cr < |x - x_0| \leq 2^{k+1} Cr\right\}$, for $k \in \mathbb{Z}_{\geq 0}$. So we split up the integral of Tf over $\mathbb{R}^n$ as sum of the integrals on the parts of the onion. Using the above observation, and the fact that f is an atom, we estimate the integral on the core as follows:

$$\int_{\tilde{B}} 1.Tf(x)dx \leq \left|\tilde{B}\right|^{1/2} \|Tf\|_2 \leq C^{n/2} |B|^{1/2} \|f\|_2 \leq C^{n/2}$$

Now to estimate the integral on each of the layers R_k, we will show that for many $\lambda \in \Lambda$, $|\alpha(\lambda)|^2 |\psi_\lambda(x)|^2 = 0$, given that $x \in R_k$. Indeed, as we've mentioned before, there exists a constant m, such that the support of ψ_λ is contained in $mQ(\lambda)$ and by hypothesis, the support of f is contained in B; we have $\alpha(\lambda) = (f, \psi_\lambda)$, so that $\alpha(\lambda)$ is zero if the intersection of $mQ(\lambda)$ with B is empty, and if the intersection of $mQ(\lambda)$ with R_k is emtpy, then $|\psi_\lambda(x)|^2 = 0$, for $x \in R_k$. So to evaluate Tf on R_k, we merely need to sum over the $\lambda \in \Lambda$ such that $mQ(\lambda) \cap B \cap R_k \neq \emptyset$. If λ satisfies this property, then $Q(\lambda)$ must be fairly big. Precisely, if we choose C sufficiently large, then there exists a positive constant c such that $2^{-j} \geq cr2^k$.

It can then be shown that $|(f, \psi_\lambda)| \leq C2^{nj/2}2^j r$, and hence that for $x \in R_k$, $Tf(x) \leq C'2^{-k(n+1)}r^{-n}$.

Then

$$\int_{R_k} Tf(x)\, dx \leq |R_k|\, C'2^{-k(n+1)r^{-n}}$$

But the volume of $|R_k|$ is the difference between the volume of a ball of radius $2^{k+1}Cr$ and that of a ball of radius $2^k Cr$, so there exists a constant D (depending on the dimension n) such that

$$\begin{aligned} |R_k| &= D\left(2^{k+1}Cr\right)^n - D\left(2^k Cr\right)^n \\ &= DC^n r^n 2^{kn}(2^n - 1) \end{aligned}$$

and we have

$$\begin{aligned} \int_{R_k} Tf(x)\, dx &\leq DC^n r^n 2^{kn}(2^n - 1)\, C'2^{-k(n+1)}r^{-n} \\ &\leq DC^n C'(2^n - 1)\, 2^{-k} \\ &= D'2^{-k} \end{aligned}$$

where $D' = DC^n C'(2^n - 1)$. Then $\sum_{k=0}^{\infty} \int_{R_k} Tf(x)\, dx \leq 2D' < \infty$. So indeed Tf is integrable if f is an atom of $H^1(\mathbb{R}^n)$.

(3)⇒(1): By what has gone before, we have proved the inequality

$$\left\| \sum_{\lambda\in\Lambda} \alpha(\lambda)\,\psi_\lambda(x) \right\|_1 \leq C \left\| \left(\sum_{\lambda\in\Lambda} |\alpha(\lambda)|^2\,|Q(\lambda)|^{-1}\,\chi_{R(\lambda)}(x) \right)^{1/2} \right\|_1$$

Replace $\alpha(\lambda)$ with $\varepsilon(\lambda)\,\alpha(\lambda)$, the right-hand remains the same, and the left hand becomes the quantity we wished to show the finiteness of.

Chapter 7

Notes and Comments

Though the wavelet concept is relatively new, a tremendous amount of literature already exists on the subject, perhaps due to its interdisciplinary nature. Many applications of the wavelet theory have been discovered, ranging from image compression methods, mathematical models of human hearing, through to fast integral transform techniques.

From a pure mathematical point of view, one of the most appealing aspects of wavelets is that they are effective in most spaces of functions and distributions. I've shown that they form an unconditional basis of $H^1(\mathbb{R}^n)$, but the same is true for the spaces $L^p(\mathbb{R}^n)$ with $1 < p < \infty$ and for the Sobolev and Hölder spaces, to name but a few.

The treatment I have presented closely follows that given in Meyer[1], which deals almost exclusively with orthonormal wavelet bases and wavelet series. The other side of the coin which I have often used but not explicitly mentioned, is the integral wavelet transform, which is to wavelet analysis what the Fourier transform is to Fourier analysis, but though Fourier series bear little relation to Fourier transforms, the coefficients of a wavelet series can be recovered from the integral wavelet transform by simply evaluating the transform at positions on an appropriate dyadic grid.

Fourier methods perform no localization, they allow you to find out *what* happens, but not *where*. This can be partially overcome by the use of heuristic methods such as the windowed Fourier transform. The ordinary Fourier transform of f is computed as the inner product of f

with the analyzing function $e^{i\omega t}$, that is

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt$$

but the windowed Fourier transform takes the inner product with a 'windowed version' of the function to give

$$(\widetilde{\mathcal{G}}_b f)(\omega) = \int_{-\infty}^{\infty} \left(e^{-i\omega t} f(t)\right) \overline{w(t-b)} dt$$

where $\overline{w(t-b)}$ is the window function. This transform gives the spectral information of the signal at time b, but is deficient in that the window function is of fixed width, giving poor frequency resolution where the signal has a low frequency, and a poor temporal resolution where the signal has a high frequency. The solution is to use the integral wavelet transform, which uses window functions of varying width $(W_\psi) f(b,a) = |a|^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt$. When ψ is a basic wavelet as discussed in the chapter on orthonormal wavelets, then we see that if $k \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, then $(W_\psi) f(k,j)$ is simply $\left(f, \psi_{j,k}\right)$.

The obvious question concerns the inversion of this transform. If the values of the transform are known for all $a, b \in \mathbb{R}$, then under a certain condition on the Fourier transform of the basic wavelet, one can recover the original function by using a very similar integral transform, indeed the kernel used is $\psi\left(\frac{t-b}{a}\right)$.When the wavelets under consideration are the orthonormal wavelets of this article, then by definition, we recover f from its wavelet series.

More general wavelets can be considered. For instance, we may require that the sequence $\left\{\psi_{j,k}\right\}$ forms a Riesz basis, rather than an orthonormal basis of $L^2\left(\mathbb{R}^n\right)$. In this case, there is a 'dual' Riesz basis $\left\{\psi^{j,k}\right\}$ such that $\left(\psi_{j,k}, \psi^{l,m}\right) = \delta_{j,l} \cdot \delta_{k,m}$, and so we can recover f by $f(x) = \sum (f, \psi_{j,k}) \psi^{j,k}(x)$. If certain restrictions are placed on ψ, then the dual basis $\left\{\psi^{j,k}\right\}$ will be generated by a dual wavelet $\tilde{\psi}$, and we have reconstruction via a wavelet series.

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Bibliography

[1] Y. Meyer, *Wavelets and operators*, Cambridge University Press, Cambridge, 1992.

[2] C. K. Chui, *An Introduction to Wavelets*, Academic Press, Inc., New York, 1992.

[3] G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, Boston, 1994.

[4] J. Diestel, H. Jarchow & A. Tonge, *Absolutely Summing Operators*, Cambridge University Press, Cambridge, 1995.

[5] Y. Meyer, *Wavelets: Algorithms and Applications*, Translated and Revised by R. D. Ryan, SIAM, Philadelphia, 1993.

[6] H. Dym & H. P. McKean, *Fourier Series and Integrals*, Academic Press, Inc., New York, 1972.

[7] I. M. Gel'fand & G. E. Shilov, *Generalized Functions*, Adademic Press Inc., New York, 1964.

[8] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Series on Applied Mathematics, SIAM, Philadelphia, 1992.

[9] S. Mallat, *Multiresolution approximation and wavelet orthonormal bases of* $L^2(\mathbb{R}^n)$, Trans. Amer. Math. Soc., **315** (1989), 69–87.

[10] B. Bollobás, *Linear Analysis, an introductory course*, Cambridge University Press, Cambridge, 1990.

[11] L. Carleson, *Convergence and growth of partial sums of Fourier series*, Acta Math., **116** (1966), 135–57.

[12] T. W. Körner, *Fourier Analysis*, Cambridge University Press, Cambridge, 1988.

[13] S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, American Mathematical Society, Providence, 1963.

[14] A. Grigoryan, *Probability Theory*, An undergraduate lecture course given at Imperial College, London, 2000.

[15] T. J. Lyons, *Complex Analysis II,* An undergraduate lecture course given at Imperial College, London, 1999.